March 13, 2025

Outline

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   - Continuous distribution
2. Integral Techniques
3. Exercises and Solutions

Probability Distributions

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  - **Continuous distribution:** uncountably infinite outcomes within intervals.

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  - 3 heads: $\{HHH\} \rightarrow$ Probability: $1/8$

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- Normalization condition:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

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- It represents the center of mass or balance point of the probability density function.

Integral Techniques: Gaussian Integral

Evaluate integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx$$

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- Differential area transformation: $dxdy = rdrd\theta$

Integral Techniques: Calculation

Integral evaluation step-by-step:

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dxdy$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r\, drd\theta$$

$$= \left(\int_{0}^{2\pi} d\theta\right)\left(\int_{0}^{\infty} e^{-r^2} r\, dr\right)$$

$$= (2\pi)\cdot\frac{1}{2} = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

Conclusion: Gaussian integral evaluates to $\sqrt{\pi}$.

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Potential questions:

- Why are integrals of odd functions zero?
- How does symmetry simplify the evaluation?

Solution: Odd Function Integral

Evaluate integral:

$$\int_{-\infty}^{\infty} x^3 e^{-x^2}\, dx$$

Step-by-step reasoning:

- Recognize integrand $x^3 e^{-x^2}$ is an odd function.

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  Therefore,

$$\int_{-\infty}^{\infty} x^3 e^{-x^2}\, dx = 0$$

Solution: Even Function Integral

Evaluate integral:

$$\int_{-\infty}^{\infty} x^4 e^{-x^2}\, dx$$

- The integrand is even: symmetric about $x = 0$, allowing simplification:

$$2\int_{0}^{\infty} x^4 e^{-x^2}\, dx$$

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- Substitute $u = x^2$; thus, $du = 2xdx$ or $xdx = du/2$, giving:

$$= 2\int_{0}^{\infty} u^2 e^{-u} \frac{du}{2\sqrt{u}} = \int_{0}^{\infty} u^{3/2} e^{-u}\, du$$

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Solution: Even Function Integral

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- Evaluate using integration by parts twice:

$$\begin{aligned}\int_0^{\infty} u^{3/2} e^{-u}\, du &= \frac{3}{2}\int_0^{\infty} u^{1/2} e^{-u}\, du \\ &= \frac{3}{2}\cdot\frac{1}{2}\int_0^{\infty} u^{-1/2} e^{-u}\, du \\ &= \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{\pi}}{1} \\ &= \frac{3\sqrt{\pi}}{4}\end{aligned}$$

Integration by Parts (Detailed Explanation)

Evaluate integral:

$$\int_0^\infty u^{3/2} e^{-u}\, du$$

Integration by parts formula:

$$\int u\, dv = uv - \int v\, du$$

- First integration:

$$u = u^{3/2}, \quad dv = e^{-u} du \Rightarrow du = \frac{3}{2} u^{1/2} du, \quad v = -e^{-u}$$

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